JOURNAL OF COMPUTATIONAL PHYSICS **86**, 56–74 (1990)

Semi-implicit Finite Difference Methods for the Two-Dimensional Shallow Water Equations

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Received March 25, 1988; Revised February 21, 1989

In this paper a semi-implicit finite difference method for the 2-dimensional shallow water equations is derived and discussed. A characteristic analysis of the governing equations is carried out first, in order to determine those terms to be discretized implicitly so that the stability of the method will not depend upon the celerity. Such terms are the gradient of the water surface elevation in the momentum equations and the velocity divergence in the continuity equation. The convective terms are discretized explicitly. The simpler explicit discretization for the convective terms is the upwind discretization which is conditionally stable and introduces some artificial viscosity. It is shown that the stability restriction is eliminated and the artificial viscosity is reduced when an Eulerian–Lagrangian approach with large time steps is used to discretize the convective terms. This method, at each time step, requires the solution of a linear, symmetric, 5-diagonal system. Such a system is diagonally dominant with positive elements on the main diagonal and negative ones elsewhere. Thus, existence and uniqueness of the numerical solution is assured. The resulting algorithm is mass conservative and fully vectorizable for an efficient implementation on modern vector computers. The performance of this method is further improved when used in combination with an ADI technique which results in two sets of simpler, linear 3-diagonal systems and maintains all the properties described above. © 1990 Academic Press, Inc.

1. INTRODUCTION

The 2-dimensional shallow water equations constitute a system of quasilinear hyperbolic partial differential equations. Such equations have the form

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-g\frac{\partial z}{\partial x}-\gamma u+\tau_x\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-g\frac{\partial z}{\partial y}-\gamma v+\tau_y\\
&\frac{\partial z}{\partial t}+\frac{\partial[(h+z)u]}{\partial x}+\frac{\partial[(h+z)v]}{\partial y}=0,
\end{aligned} \tag{1}$$

0021-9991/90 $3.00

where $u(x, y, t)$ and $v(x, y, t)$ are the depth-averaged velocity components in the x and in the y directions, respectively, $z(x, y, t)$ is the water surface elevation measured from the undisturbed water surface, $h(x, y)$ is the water depth also measured from the undisturbed water surface, g is the constant gravitational acceleration, τ_x and τ_y are wind stress terms in the x and y directions, respectively, and γ is the bottom friction coefficient. Typically, γ is given by

$$\gamma = \frac{g\sqrt{u^2+v^2}}{C_z^2 H}, \tag{2}$$

where $H(x, y, t) = h(x, y) + z(x, y, t)$ is the total water depth, and C_z is the Chezy friction coefficient.

Several numerical methods for Eq. (1) are known in the current literature and are now widely used (see, e.g., [1, 2, 6–8, 11, 12, 16, 17]). If the solution of these equations is expected to have sharp gradients, the numerical solution may either develop spurious oscillations or be affected by a large artificial viscosity. Spurious oscillations or artificial viscosity often destroy the accuracy so that the numerical solution simply becomes unacceptable [3]. Another severe limitation of standard explicit numerical methods for Eqs. (1) is due to the stability restriction imposed by the Courant–Friedrich–Lewy condition. This restriction usually requires a much smaller time step than permitted by accuracy considerations. Of course, a fully implicit discretization of the governing equations often leads to methods which are unconditionally stable. Fully implicit methods, however, involve the simultaneous solution of a large number of coupled nonlinear equations. Moreover, for accuracy, the time step cannot be arbitrarily large so that these methods often become impractical.

A very popular numerical method for solving Eqs. (1) is the one developed by Leendertse [11, 12]. The numerical results obtained from the integration of these equations are also used by simulation models to calculate residual currents [6], and to analyze the salinity distribution in well-mixed estuaries [7]. Leendertse's method uses a staggered grid and a semi-implicit approach to discretize all terms in the governing equations. With this method an alternating direction implicit (ADI) technique is used. Thus, if the computational region is divided into $n \times m$ mesh points, at each time step, it yields n linear tridiagonal systems of $2m$ equations in $2m$ unknowns in one direction, and m linear tridiagonal systems of $2n$ equations in $2n$ unknowns in the other spatial direction. Leendertse's method successfully uses a semi-implicit type of discretization which results in a practical algorithm.

The method which will be derived next, uses a space staggered mesh on which the governing equations are discretized with a semi-implicit technique in such a way that the stability of the method does not depend upon the celerity $\sqrt{gH}$. The convective terms can be discretized by using a simple, but highly dissipative upwind formula or by using a more accurate Eulerian–Lagrangian approach. In the latter case the resulting algorithm is also shown to be unconditionally stable thus permitting the use of larger steps with corresponding improvements in both efficiency and accuracy. Computationally, at each time step, we first derive a $n \times m$ linear 5-diagonal system where the new water surface elevation is the only unknown. Such a system is symmetric and positive definite. Thus it can be solved uniquely and efficiently by using a preconditioned conjugate gradient method. Then, the fluid velocity is obtained explicitly from the discretized momentum equations. Overall, most of the required arithmetic operations can be made independent from each other and highly vectorizable for an efficient implementation on vector computers.

The efficiency of these methods can be further improved by introducing a two time level approach which alternatively solves for the x-momentum and continuity equation in a first level, and for the y-momentum and continuity equation in the second level. The overall computational effort is thus reduced since the 2-dimensional system is so decomposed into n tridiagonal systems of m equations in m unknowns in the first level, and m tridiagonal systems of n equations in n unknowns in the second level. Each of these systems is linear, strictly diagonally dominant, and has positive elements on the main diagonal and negative ones elsewhere. Thus the existence and the uniqueness of a stable numerical solution is always assured.

2. Characteristic Analysis of the Governing Equations

Equations (1) form a quasilinear hyperbolic system of partial differential equations in three independent variables. In order to determine the particular semi-implicit discretization, whose stability is independent on the celerity, we will first analyze the characteristic cone of the governing equations (see [4]). To this purpose let us rewrite Eqs. (1) in the equivalent form

$$\begin{gathered}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial z}{\partial x}=-\gamma u+\tau_x \\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial z}{\partial y}=-\gamma v+\tau_y \\
\frac{\partial z}{\partial t}+u\frac{\partial z}{\partial x}+v\frac{\partial z}{\partial y}+H\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]=-u\frac{\partial h}{\partial x}-v\frac{\partial h}{\partial y},
\end{gathered} \tag{3}$$

or, in matrix notation,

$$\frac{\partial \mathbf{W}}{\partial t}+\mathbf{A}(\mathbf{W})\frac{\partial \mathbf{W}}{\partial x}+\mathbf{B}(\mathbf{W})\frac{\partial \mathbf{W}}{\partial y}=\mathbf{D}(\mathbf{W}), \tag{4}$$

where $\mathbf{W}=(u, v, z)^{\mathrm{T}}$, and

$$\mathbf{A}=\begin{bmatrix} u & 0 & g \\ 0 & u & 0 \\ H & 0 & u \end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix} v & 0 & 0 \\ 0 & v & g \\ 0 & H & v \end{bmatrix}, \qquad \mathbf{D}=\begin{bmatrix} -\gamma u+\tau_x \\ -\gamma v+\tau_y \\ -u\dfrac{\partial h}{\partial x}-v\dfrac{\partial h}{\partial y} \end{bmatrix}.$$

If $\mathbf{I}$ denotes the identity matrix, the characteristic equation of system (4) is given by

$$\det(q\mathbf{I} + r\mathbf{A} + s\mathbf{B}) = 0; \tag{5}$$

that is,

$$(q + ru + sv)[(q + ru + sv)^2 - gH(r^2 + s^2)] = 0. \tag{6}$$

The triples (q, r, s) satisfying Eq. (6) are the directions normal to the characteristic cone at its vertex [10]. Equation (6) decomposes into the two equations

$$q + ru + sv = 0, \tag{7}$$

and

$$(q + ru + sv)^2 - gH(r^2 + s^2) = 0. \tag{8}$$

Hence, as shown in Fig. 1, the local characteristic cone with vertex in (x_0, y_0, t_0) consists of the line through (x_0, y_0, t_0) parallel to the vector $(1, u, v)$, and the cone whose equation is

$$[(x - x_0) - u(t - t_0)]^2 + [(y - y_0) - v(t - t_0)]^2 - gH(t - t_0)^2 = 0; \tag{9}$$

in fact, on the cone surface, the gradient of the left-hand side of Eq. (9) satisfies Eq. (8).

Note that, whereas the first part of the characteristic cone depends only on the fluid velocity u and v, the second part, which is defined by Eq. (8), depends also upon the celerity $\sqrt{gH}$. Note also that the term gH in Eq. (6) arises from the off-diagonal terms g and H in the matrices $\mathbf{A}$ and $\mathbf{B}$. These are the coefficient of $\partial z/\partial x$ in the first equation, the coefficient of $\partial z/\partial y$ in the second equation, and the coefficient of $\partial u/\partial x$ and $\partial v/\partial y$ in the third equation of system (3). Consequently, these derivatives must be discretized implicitly in order for the stability of the method to be independent of the celerity.

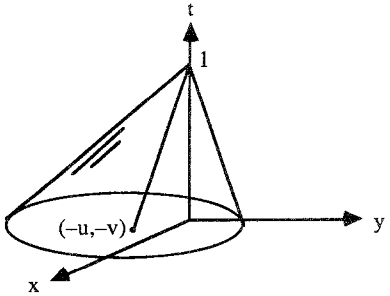

FIG. 1. Characteristic cone through (0, 0, 1).

3. A Semi-Implicit Numerical Method

Based on the discussion of the previous section we will derive, next, a numerical method for Eq. (3) in which the gradient of surface elevation in the momentum equations and the velocity divergence in the continuity equation will be discretized implicitly. The convective terms in the momentum equations, however, will be discretized explicitly. As it concerns the right-hand sides of Eqs. (3) we will proceed as follows. For stability, the friction terms in the momentum equations will be discretized implicitly, but the friction coefficient γ will be evaluated explicitly so that the resulting algebraic system to be solved will be linear. The continuity equation will be considered in its original conservative form as

$$\frac{\partial z}{\partial t}+\frac{\partial[(h+z)u]}{\partial x}+\frac{\partial[(h+z)v]}{\partial y}=0,$$

where u and v will be discretized implicitly, while the total water depth $H=h+z$ will be taken explicitly.

Next, as shown in Fig. 2, we introduce a spatial mesh which consists of rectangular cells of length Δx and width Δy. Each cell is numbered at its center with indices i and j. The discrete u velocity is then defined at half integer i and integer j; v is defined at integer i and half integer j, and z is defined at integer i and integer j. The water depth $h(x, y)$ is assumed to be known throughout. Then, a general semi-implicit discretization of Eqs. (3) takes the form

$$\begin{aligned}
u^{k+1}_{i+1/2,j} &= Fu^{k}_{i+1/2,j} - g\frac{\Delta t}{\Delta x}(z^{k+1}_{i+1,j} - z^{k+1}_{i,j}) \\
&\quad - \Delta t(\gamma^{k}_{i+1/2,j}u^{k+1}_{i+1/2,j} - \tau_x) \\
v^{k+1}_{i,j+1/2} &= Fv^{k}_{i,j+1/2} - g\frac{\Delta t}{\Delta y}(z^{k+1}_{i,j+1} - z^{k+1}_{i,j}) \\
&\quad - \Delta t(\gamma^{k}_{i,j+1/2}v^{k+1}_{i,j+1/2} - \tau_y) \\
z^{k+1}_{i,j} &= z^{k}_{i,j} - \frac{\Delta t}{\Delta x}[(\bar{z}^{k}_{i+1/2,j} + h_{i+1/2,j})\,u^{k+1}_{i+1/2,j} \\
&\quad - (\bar{z}^{k}_{i-1/2,j} + h_{i-1/2,j})\,u^{k+1}_{i-1/2,j}] \\
&\quad - \frac{\Delta t}{\Delta y}[(\bar{z}^{k}_{i,j+1/2} + h_{i,j+1/2})\,v^{k+1}_{i,j+1/2} \\
&\quad - (\bar{z}^{k}_{i,j-1/2} + h_{i,j-1/2})\,v^{k+1}_{i,j-1/2}]
\end{aligned} \tag{10}$$

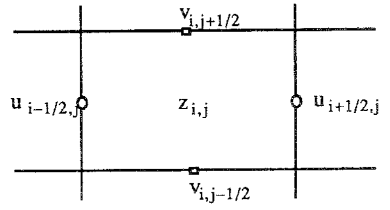

FIG. 2. Spatial mesh.

or, equivalently,

$$
\begin{aligned}
(1+\gamma^k_{i+1/2,j}\Delta t)\, u^{k+1}_{i+1/2,j} &= Fu^k_{i+1/2,j} \\
&\qquad - g\frac{\Delta t}{\Delta x}(z^{k+1}_{i+1,j}-z^{k+1}_{i,j})+\Delta t\tau_x \\
(1+\gamma^k_{i,j+1/2}\Delta t)\, v^{k+1}_{i,j+1/2} &= Fv^k_{i,j+1/2} \\
&\qquad - g\frac{\Delta t}{\Delta y}(z^{k+1}_{i,j+1}-z^{k+1}_{i,j})+\Delta t\tau_y \\
z^{k+1}_{i,j} = z^k_{i,j} - \frac{\Delta t}{\Delta x}&[(\bar z^k_{i+1/2,j}+h_{i+1/2,j})\, u^{k+1}_{i+1/2,j} \\
&- (\bar z^k_{i-1/2,j}+h_{i-1/2,j})\, u^{k+1}_{i-1/2,j}] \\
&- \frac{\Delta t}{\Delta y}[(\bar z^k_{i,j+1/2}+h_{i,j+1/2})\, v^{k+1}_{i,j+1/2} - (\bar z^k_{i,j-1/2}+h_{i,j-1/2})\, v^{k+1}_{i,j-1/2}],
\end{aligned}
\tag{11}
$$

where $\bar z^k_{i\pm 1/2,j}$ and $\bar z^k_{i,j\pm 1/2}$ are defined as simple averages from the closest scalar grid points, that is,

$$\bar z^k_{i\pm 1/2,j} = \tfrac{1}{2}(z^k_{i,j}+z^k_{i\pm 1,j}), \qquad \bar z^k_{i,j\pm 1/2} = \tfrac{1}{2}(z^k_{i,j}+z^k_{i,j\pm 1}).$$

In (11) F is an explicit, nonlinear finite difference operator, corresponding to the spatial discretization of the convective terms $u_t+uu_x+vu_y$ and $v_t+uv_x+vv_y$. A particular form for F can be chosen in a variety of ways and will be analyzed later.

For any structure given to F, Eqs. (11) constitute a linear system of equations with unknowns $u^{k+1}_{i+1/2,j}$, $v^{k+1}_{i,j+1/2}$, and $z^{k+1}_{i,j}$ over the entire cell configuration. This system has to be solved at each time step to determine, recursively, values of the field variables from given initial data. From a computational point of view, since most of the computer time will be devoted to the solution of system (11), we will first reduce this system to a smaller one in which $z^{k+1}_{i,j}$ are the only unknowns. Specifically, substitution of the expressions for $u^{k+1}_{i\pm 1/2,j}$ and $v^{k+1}_{i,j\pm 1/2}$ from the first two equations into the third equation of system (11) yields

$$
\begin{aligned}
z_{i,j}^{k+1} &- g\frac{\Delta t^2}{\Delta x^2}\left[\frac{\bar{z}_{i+1/2,j}^k + h_{i+1/2,j}}{1+\gamma_{i+1/2,j}^k \Delta t}(z_{i+1,j}^{k+1} - z_{i,j}^{k+1})\right.\\
&\left. - \frac{\bar{z}_{i-1/2,j}^k + h_{i-1/2,j}}{1+\gamma_{i-1/2,j}^k \Delta t}(z_{i,j}^{k+1} - z_{i-1,j}^{k+1})\right]\\
&- g\frac{\Delta t^2}{\Delta y^2}\left[\frac{\bar{z}_{i,j+1/2}^k + h_{i,j+1/2}}{1+\gamma_{i,j+1/2}^k \Delta t}(z_{i,j+1}^{k+1} - z_{i,j}^{k+1})\right.\\
&\left. - \frac{\bar{z}_{i,j-1/2}^k + h_{i,j-1/2}}{1+\gamma_{i,j-1/2}^k \Delta t}(z_{i,j}^{k+1} - z_{i,j-1}^{k+1})\right]\\
= z_{i,j}^k &- \frac{\Delta t}{\Delta x}\left[\frac{\bar{z}_{i+1/2,j}^k + h_{i+1/2,j}}{1+\gamma_{i+1/2,j}^k \Delta t}(Fu_{i+1/2,j}^k + \Delta t\tau_x)\right.\\
&\left. - \frac{\bar{z}_{i-1/2,j}^k + h_{i-1/2,j}}{1+\gamma_{i-1/2,j}^k \Delta t}(Fu_{i-1/2,j}^k + \Delta t\tau_x)\right]\\
&- \frac{\Delta t}{\Delta y}\left[\frac{\bar{z}_{i,j+1/2}^k + h_{i,j+1/2}}{1+\gamma_{i,j+1/2}^k \Delta t}(Fv_{i,j+1/2}^k + \Delta t\tau_y)\right.\\
&\left. - \frac{\bar{z}_{i,j-1/2}^k + h_{i,j-1/2}}{1+\gamma_{i,j-1/2}^k \Delta t}(Fv_{i,j-1/2}^k + \Delta t\tau_y)\right]. \qquad (12)
\end{aligned}
$$

Equations (12) constitute a linear 5-diagonal system of equations for $z_{i,j}^{k+1}$. This system, under the assumption $(\bar{z}^k + h)_{i\pm 1/2,j} > 0$ and $(\bar{z}^k + h)_{i,j\pm 1/2} > 0$, is symmetric and strictly diagonally dominant with positive elements on the main diagonal and negative ones elsewhere. Thus, it is positive definite and has a unique solution. In practice, this 5-diagonal system can be solved very efficiently by preconditioned conjugate gradient methods which, in combination with multicoloring techniques, are suitable for vector computations (see Ref. [14] for further details).

4. Stability of the Method

The stability analysis of the semi-implicit method (11) will be carried out by using the von Neumann method under the assumption that our differential equations (3) are linear and defined on an infinite spatial domain, or with periodic boundary conditions on a finite domain. Hence, the difference equations (11) reduce to

$$
\begin{aligned}
&(1+\gamma\Delta t)\,u_{i+1/2,j}^{k+1} + g\frac{\Delta t}{\Delta x}(z_{i+1,j}^{k+1} - z_{i,j}^{k+1}) = Fu_{i+1/2,j}^k\\
&(1+\gamma\Delta t)\,v_{i,j+1/2}^{k+1} + g\frac{\Delta t}{\Delta y}(z_{i,j+1}^{k+1} - z_{i,j}^{k+1}) = Fv_{i,j+1/2}^k\\
&z_{i,j}^{k+1} + H\frac{\Delta t}{\Delta x}(u_{i+1/2,j}^{k+1} - u_{i-1/2,j}^{k+1})\\
&\qquad + H\frac{\Delta t}{\Delta y}(v_{i,j+1/2}^{k+1} - v_{i,j-1/2}^{k+1}) = z_{i,j}^k,
\end{aligned}
\qquad (13)
$$

where the operator F has been assumed to be linear, and all the coefficients have been assumed to be constants.

Now, by changing variables u and v with $U=u\sqrt{1+\gamma\Delta t}$ and $V=v\sqrt{1+\gamma\Delta t}$, respectively, and variable z with variable $Z=z\sqrt{g/H}$, Eqs. (13) become

$$\begin{aligned} &U^{k+1}_{i+1/2,j}+\frac{\Delta t\sqrt{gH}}{\Delta x\sqrt{1+\gamma\Delta t}}(Z^{k+1}_{i+1,j}-Z^{k+1}_{i,j})=\frac{FU^k_{i+1/2,j}}{1+\gamma\Delta t}\\ &V^{k+1}_{i,j+1/2}+\frac{\Delta t\sqrt{gH}}{\Delta x\sqrt{1+\gamma\Delta t}}(Z^{k+1}_{i,j+1}-Z^{k+1}_{i,j})=\frac{FV^k_{i,j+1/2}}{1+\gamma\Delta t}\\ &Z^{k+1}_{i,j}+\frac{\Delta t\sqrt{gH}}{\Delta x\sqrt{1+\gamma\Delta t}}(U^{k+1}_{i+1/2,j}-U^{k+1}_{i-1/2,j})\\ &\quad+\frac{\Delta t\sqrt{gH}}{\Delta x\sqrt{1+\gamma\Delta t}}(V^{k+1}_{i,j+1/2}-V^{k+1}_{i,j-1/2})=Z^k_{i,j}. \end{aligned} \tag{14}$$

In order to analyze the stability of Eqs. (14) with the von Neumann method, a Fourier mode is introduced for each field variable U, V, and Z and the stability analysis is carried out on the corresponding amplitude functions. Specifically, $U^k_{i+1/2,j}$, $V^k_{i,j+1/2}$, and $Z^k_{i,j}$ are replaced in (14) by $\hat{U}^k e^{I[(i+1/2)\alpha+j\beta]}$, $\hat{V}^k e^{I[i\alpha+(j+1/2)\beta]}$, and $\hat{Z}^k e^{I(i\alpha+j\beta)}$, respectively, where $\hat{U}^k$, $\hat{V}^k$, and $\hat{Z}^k$ are the amplitude functions of U, V, and Z at time level t_k, $I=\sqrt{-1}$, and α and β are the x and the y phase angles. Thus, after some simplifications, (14) implies

$$\begin{aligned} &\hat{U}^{k+1}+\frac{\Delta t\sqrt{gH}}{\Delta x\sqrt{1+\gamma\Delta t}}(e^{I\alpha/2}-e^{-I\alpha/2})\,\hat{Z}^{k+1}=\frac{f\hat{U}^k}{1+\gamma\Delta t}\\ &\hat{V}^{k+1}+\frac{\Delta t\sqrt{gH}}{\Delta y\sqrt{1+\gamma\Delta t}}(e^{I\beta/2}-e^{-I\beta/2})\,\hat{Z}^{k+1}=\frac{f\hat{V}^k}{1+\gamma\Delta t}\\ &\hat{Z}^{k+1}+\frac{\Delta t\sqrt{gH}}{\Delta x\sqrt{1+\gamma\Delta t}}(e^{I\alpha/2}-e^{-I\alpha/2})\,\hat{U}^{k+1}\\ &\quad+\frac{\Delta t\sqrt{gH}}{\Delta y\sqrt{1+\gamma\Delta t}}(e^{I\beta/2}-e^{-I\beta/2})\,\hat{V}^{k+1}=\hat{Z}^k, \end{aligned} \tag{15}$$

where f is the amplification factor of the linearized operator F which will be analyzed later. Now, since $e^{I\alpha/2}-e^{-I\alpha/2}=2I\sin(\alpha/2)$, Eqs. (15) become

$$\begin{aligned} &\hat{U}^{k+1}+2Ip\hat{Z}^{k+1}=\frac{f\hat{U}^k}{1+\gamma\Delta t}\\ &\hat{V}^{k+1}+2Iq\hat{Z}^{k+1}=\frac{f\hat{V}^k}{1+\gamma\Delta t}\\ &\hat{Z}^{k+1}+2Ip\hat{U}^{k+1}+2Iq\hat{V}^{k+1}=\hat{Z}^k, \end{aligned} \tag{16}$$

where

$$p=\frac{\Delta t\sqrt{gH}}{\Delta x\sqrt{1+\gamma\Delta t}}\sin\left(\frac{\alpha}{2}\right), \qquad q=\frac{\Delta t\sqrt{gH}}{\Delta y\sqrt{1+\gamma\Delta t}}\sin\left(\frac{\beta}{2}\right). \tag{17}$$

In matrix notation, Eqs. (16) can be written as

$$\mathbf{R}\hat{\mathbf{W}}^{k+1}=\mathbf{S}\hat{\mathbf{W}}^{k}, \tag{18}$$

where $\hat{\mathbf{W}}^k=[\hat{U}^k, \hat{V}^k, \hat{Z}^k]^{\mathrm{T}}$,

$$\mathbf{R}=\begin{bmatrix} 1 & 0 & 2Ip \\ 0 & 1 & 2Iq \\ 2Ip & 2Iq & 1 \end{bmatrix}, \qquad \mathbf{S}=\begin{bmatrix} \dfrac{f}{1+\gamma\Delta t} & 0 & 0 \\ 0 & \dfrac{f}{1+\gamma\Delta t} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, the amplification matrix of the method is $\mathbf{G}=\mathbf{R}^{-1}\mathbf{S}$, and a necessary and sufficient condition for stability is $\|\mathbf{G}\|_2\leqslant 1$ identically for every α and β. But, since $\|\mathbf{G}\|\leqslant\|\mathbf{R}^{-1}\|\cdot\|\mathbf{S}\|$, we are seeking the conditions for which $\|\mathbf{R}^{-1}\|\leqslant 1$ and $\|\mathbf{S}\|\leqslant 1$.

Note now, that the two matrices **R** and **S**, and hence also $\mathbf{R}^{-1}$, are normal matrices; that is, they commute with their respective hermitian conjugate. Thus, the norms of $\mathbf{R}^{-1}$ and of **S** are equal to their respective spectral radius. But, the eigenvalues of **R** are

$$\lambda_{\mathbf{R}}=1+2I\sqrt{p^2+q^2}, \quad 1, \quad 1-2I\sqrt{p^2+q^2}, \tag{19}$$

thus the spectral radius of $\mathbf{R}^{-1}$ is always no greater than unity. Next, the eigenvalues of **S** are

$$\lambda_{\mathbf{S}}=\frac{f}{1+\gamma\Delta t}, \quad 1, \quad \frac{f}{1+\gamma\Delta t}. \tag{20}$$

Hence, in order for spectral radius of **S** not to exceed unity, it is sufficient that

$$|f|\leqslant 1, \tag{21}$$

identically for every α and β. Thus the stability, the consistency, and the accuracy of a finite difference method (11), depends only on the choice of the difference operator F.

5. Convective Terms Discretization

As it concerns the choice for the difference operator F, here we will consider the two cases, where a simple upwind differencing approximation is used and the case where a more flexible Eulerian–Lagrangian approach is used. In the first case, the first-order spatial derivatives are replaced by first-order difference quotiens, backward if u(or v) is positive and forward if u (or v) is negative. If we assume u and v positive, F is defined as

$$Fw_{i,j}^k = w_{i,j}^k - u_{i,j}^k \frac{\Delta t}{\Delta x}(w_{i,j}^k - w_{i-1,j}^k) - v_{i,j}^k \frac{\Delta t}{\Delta y}(w_{i,j}^k - w_{i,j-1}^k), \tag{22}$$

where the convective coefficients $u_{i,j}^k$ and $v_{i,j}^k$, if not defined at (i, j), are approximated by simple averages from the closest surrounding mesh points. In this case, the amplification factor f can be determined by assuming that u and v are constants so that the linearized upwind operator F takes the form

$$Fw_{i,j}^k = w_{i,j}^k - u \frac{\Delta t}{\Delta x}(w_{i,j}^k - w_{i-1,j}^k) - v \frac{\Delta t}{\Delta y}(w_{i,j}^k - w_{i,j-1}^k). \tag{23}$$

Upon substitution of the Fourier mode in (23) we get

$$f = 1 - u\frac{\Delta t}{\Delta x} - v\frac{\Delta t}{\Delta y} + u\frac{\Delta t}{\Delta x}[\cos(\alpha) - I\sin(\alpha)] + v\frac{\Delta t}{\Delta y}[\cos(\beta) - I\sin(\beta)], \tag{24}$$

so that the modulus of f satisfies the following inequality

$$|f| \leqslant \left|1 - \left(u\frac{\Delta t}{\Delta x} + v\frac{\Delta t}{\Delta y}\right)\right| + \left|u\frac{\Delta t}{\Delta x}\right| + \left|v\frac{\Delta t}{\Delta y}\right|. \tag{25}$$

Hence, since u and v have been assumed to be positive, the condition on the time step to have $|f| \leqslant 1$ is $1 - [u(\Delta t/\Delta x) + v(\Delta t/\Delta y)] \geqslant 0$, which, for more general coefficients u and v, can be written as

$$\Delta t \leqslant \left[\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y}\right]^{-1}. \tag{26}$$

Note, however, that for large $|u|$ or $|v|$, inequality (26) becomes quite restrictive.

In order to obtain an explicit form for F which is relatively accurate and unconditionally stable, an Eulerian–Lagrangian approximation must be used (see, e.g., [3, 6, 9, 15]). To this purpose note that the convective terms can be rewritten more compactly as Lagrangian derivatives as

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}, \tag{27}$$

where the substantial derivative d/dt indicates that the time rate of change is calculated along the streamline defined by

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v. \tag{28}$$

Now, by denoting with a and b the Courant numbers

$$a = u\frac{\Delta t}{\Delta x}, \qquad b = v\frac{\Delta t}{\Delta y}, \tag{29}$$

Eq. (27) implies that the correct physical expression for $Fw^k_{i,j}$ is

$$Fw^k_{i,j} = w^k_{i-a,j-b}. \tag{30}$$

Note that $Fw^k_{i,j}$ is the value of w at time t_k in $(i-a, j-b)$ which is being convected in (i, j) in a lapsed time Δt. In general, however, a and b are not integers, and therefore $(i-a, j-b)$ is not a grid point. For this reason an interpolation formula must be used to approximate the right-hand side of Eq. (30). When u and v are positive, if $w^k_{i-a,j-b}$ is approximated with a linear interpolation between (i, j), $(i-1, j)$, and $(i, j-1)$, there results again the upwind formula (23). In this case the stability restriction (26) can be regarded as a condition that $w^k_{i-a,j-b}$ must be evaluated as an interpolation rather than extrapolation.

The Eulerian–Lagrangian methods use a generalization of the interpolation concept of $w^k_{i-a,j-b}$ between three or more mesh points which do not necessarily include the point (i, j) (see [3]). Here we will consider the case that $w^k_{i-a,j-b}$ is approximated with a bilinear interpolation over the four surrounding mesh points. To this purpose, see Fig. 3, let $a = n + p$, $b = m + q$, where n and m are integers and $0 \leqslant p < 1$, $0 \leqslant q < 1$. Then the right-hand side of (30) is approximated by

$$\begin{aligned} Fw^k_{i,j} = {} & (1-p)[(1-q)\, w^k_{i-n,j-m} + q w^k_{i-n,j-m-1}] \\ & + p[(1-q)\, w^k_{i-n-1,j-m} + q w^k_{i-n-1,j-m-1}]. \end{aligned} \tag{31}$$

In practice, since u and v are not constants, the correct value of a and b can be found from the solution of the ordinary differential equations (28). Specifically, since u and v are known only at time level t_k, we will first assume that u and v do not vary over a time step. Then, at each mesh point (i, j) Eqs. (28) will be integrated numerically from t_{k+1} to t_k by using, for instance, the Euler method.

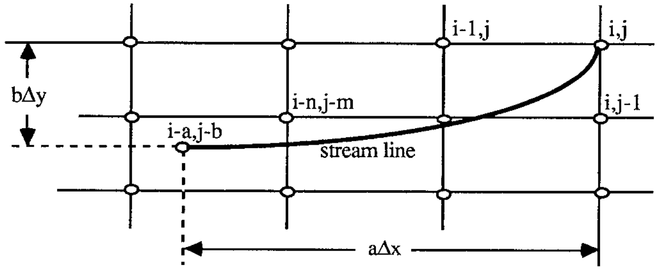

FIG. 3. The Eulerian–Lagrangian mesh.

Thus, the time step Δt is divided into N equal parts of lengths $\tau = \Delta t/N$ and Eqs. (28) are discretized backward as

$$\begin{aligned} x^{s-1} &= x^s - \tau u^k(x^s, y^s), \qquad x^N = x_i, \\ y^{s-1} &= y^s - \tau v^k(x^s, y^s), \qquad y^N = y_j, \qquad s = N, N-1, N-2, ..., 2, 1, \end{aligned} \tag{32}$$

where $u^k(x^s, y^s)$ and $v^k(x^s, y^s)$ are interpolated with a formula similar to (31). Then, at (x_i, y_j), a and b are defined by

$$a = \frac{x_i - x^0}{\Delta x}, \qquad b = \frac{y_j - y^0}{\Delta y}.$$

In so doing, the stream lines, which in general are not straight lines, are better approximated. This integration process is relatively fast expecially if performed on a vector machine since it, again, is fully vectorizable.

The amplification factor of F, when the Eulerian–Lagrangian formula (31) is used, for u and v constants, is given by

$$\begin{aligned} f &= [\cos(n\alpha) - I\sin(n\alpha)][\cos(m\beta) - I\sin(m\beta)] \\ &\quad \times [1 - p + p\cos(\alpha) - I\,p\sin(\alpha)][1 - q + q\cos(\beta) - I\,q\sin(\beta)]. \end{aligned} \tag{33}$$

Hence, since $0 \leqslant p < 1$, and $0 \leqslant q < 1$, we get $|f| \leqslant 1$ identically for every α and β and with no restriction on the time step. Thus, a first advantage of an Eulerian–Lagrangian approach to discretize the convective terms is that the resulting difference equations (11) are unconditionally stable. For consistency, however, a restriction on the time subdivision τ must be imposed in order for the approximated stream lines not to cross the solid boundaries. Specifically, the Courant numbers based on τ and the fluid velocity should not exceed unity, that is,

$$\tau \leqslant \min\left[\frac{\Delta x}{\max_{i,j}|u^k_{i+1/2,j}|}, \frac{\Delta y}{\max_{i,j}|v^k_{i,j+1/2}|}\right]. \tag{34}$$

Note that inequality (34) is sufficient to assure that the stream lines approximated by (32) will not cross the solid boundaries. Assume, for example, that the line $x = 0$ is a solid boundary and that (x^s, y^s), $x^s > 0$ is a point that lies inside the computational domain. If $x^s \geqslant \Delta x$ then, Eq. (32) and inequality (34) imply $|x^{s-1} - x^s| \leqslant \Delta x$ and hence $x^{s-1} \geqslant 0$. If $x^s < \Delta x$ then, since $u^k_{i-1/2,j} = 0$ for all j, from the first equation (32) one obtains

$$\begin{aligned} x^{s-1} &= x^s - \tau u^k(x^s, y^s) = x^s - \frac{\tau}{\Delta x}[(1 - x^s)\,u^k(0, y^s) + x^s u^k(\Delta x, y^s)] \\ &= x^s\left[1 - \frac{\tau}{\Delta x}u^k(\Delta x, y^s)\right] \geqslant 0. \end{aligned} \tag{35}$$

Thus, in no case will the stream line approximated by (x^s, y^s), $s = 0, 1, 2, ..., N$, cross the solid boundary $x = 0$.

Note that the consistency condition (34) is not required for the stability of the method but contributes to improve its accuracy. From a computational point of view it must be observed that the Eulerian–Lagrangian approach, permitting a larger step, allows less frequent solutions of linear system (12) which has to be solved each Δt time units. Within each time step, N applications of the simple Lagrangian formula (32) are required in order to approximate the streamlines through each mesh point.

6. Artificial Viscosity

The semi-implicit finite difference scheme (10) will introduce some artificial viscosity when either the upwind formula (22) or the Eulerian–Lagrangian formula (31) is used for F. In order to analyze the artificial viscosity introduced by these methods, recall that the Lagrangian derivative (27) has been discretized as

$$\frac{dw}{dt} \approx \frac{w_{i,j}^{k+1} - w_{i-a,j-b}^{k}}{\Delta t}, \tag{36}$$

where $w_{i-a,j-b}^{k}$ has been approximated either with the linear interpolation (22) or with the bilinear interpolation (31). Consider, first, the case that the upwind formula (22) is used. In this case, a Taylor series expansion about $(i-a, j-b)$ of each term in (22) yields

$$\frac{w_{i,j}^{k+1} - w_{i-a,j-b}^{k}}{\Delta t} - \frac{dw}{dt} = \frac{1}{2\Delta t}\left[\Delta x^2 a(1-a)\frac{\partial^2 w}{\partial x^2} - \Delta x \Delta y ab\frac{\partial^2 w}{\partial x\, \partial y} + \Delta y^2 b(1-b)\frac{\partial^2 w}{\partial y^2}\right] + \text{HOT}, \tag{37}$$

where HOT stands for higher order terms. The right-hand side of (37) represents the truncation error whose lowest order term has a form of a viscosity. Such an artificial viscosity is directionally dependent and, of course, has a smearing effect on the numerical solution. Note also that, due to the limitation (26) on the time step, the artificial viscosity coefficients cannot be arbitrarily reduced in this case.

When the Eulerian–Lagrangian formula (31) is used for F a Taylor series expansion about $(i-a, j-b)$ yields

$$\frac{w_{i,j}^{k+1} - w_{i-a,j-b}^{k}}{\Delta t} - \frac{dw}{dt} = \frac{1}{2\Delta t}\left[\Delta x^2 p(1-p)\frac{\partial^2 w}{\partial x^2} + \Delta y^2 q(1-q)\frac{\partial^2 w}{\partial y^2}\right] + \text{HOT}. \tag{38}$$

Note, first, that the mixed derivatives term does not appear on the right-hand side of (38). Moreover, since p and q are the decimal parts of a and b, respectively, the artificial diffusion given by (38) can be arbitrarily reduced with respect to the one given by (37). This reduction can be obtained by increasing a and b, which, since the method is unconditionally stable, can be achieved either by increasing Δt or by reducing Δx and Δy (see Ref. [3] for further details).

Complete elimination of the numerical diffusion can be achieved by using a higher order interpolation formula to define $Fw^k_{i,j}$. For instance, if a biquadratic interpolation formula over nine surrounding mesh points is used, the resulting finite difference method is entirely free from artificial viscosity. The accuracy of the method, in such a case, would not be greatly improved since biquadratic interpolation may introduce spurious oscillations (see [3]).

7. Alternating Direction Semi-Implicit

In order to simplify the solution algorithm even further, we will introduce, next, an alternating direction algorithm so that the original scheme will be decomposed into a 2-levels method involving the solution of a set of simple tridiagonal systems.

In the first level the u-velocity is assumed to be known at time $t_{k-1/2}$, and the v-velocity and the surface elevation are assumed to be known at time t_k. Then, the x-momentum and the continuity equations are finite differenced as

$$(1+\gamma^k_{i+1/2,j}\Delta t)\, u^{k+1/2}_{i+1/2,j} = Fu^{k-1/2}_{i+1/2,j} - g\frac{\Delta t}{\Delta x}(z^{k+1/2}_{i+1,j} - z^{k+1/2}_{i,j}) + \Delta t\tau_x$$

$$\begin{aligned} z^{k+1/2}_{i,j} = z^k_{i,j} &- \frac{\Delta t}{2\Delta x}\,[(\bar{z}^k_{i+1/2,j} + h_{i+1/2,j})\, u^{k+1/2}_{i+1/2,j} \\ &- (\bar{z}^k_{i-1/2,j} + h_{i-1/2,j})\, u^{k+1/2}_{i-1/2,j}] \\ &- \frac{\Delta t}{2\Delta y}\,[(\bar{z}^k_{i,j+1/2} + h_{i,j+1/2})\, v^k_{i,j+1/2} \\ &- (\bar{z}^k_{i,j-1/2} + h_{i,j-1/2})\, v^k_{i,j-1/2}], \end{aligned} \tag{39}$$

Eliminaton of $u^{k+1/2}_{i\pm 1/2,j}$ in (39) yields

$$\begin{aligned} z^{k+1/2}_{i,j} &- g\frac{\Delta t^2}{2\Delta x^2}\left[\frac{\bar{z}^k_{i+1/2,j} + h_{i+1/2,j}}{1+\gamma^k_{i+1/2,j}\Delta t}(z^{k+1/2}_{i+1,j} - z^{k+1/2}_{i,j})\right. \\ &\left. - \frac{\bar{z}^k_{i-1/2,j} + h_{i-1/2,j}}{1+\gamma^k_{i-1/2,j}\Delta t}(z^{k+1/2}_{i,j} - z^{k+1/2}_{i-1,j})\right] \\ &= z^k_{i,j} - \frac{\Delta t}{2\Delta y}\,[(\bar{z}^k_{i,j+1/2} + h_{i,j+1/2})\, v^k_{i,j+1/2} \\ &- (\bar{z}^k_{i,j-1/2} + h_{i,j-1/2})\, v^k_{i,j-1/2}] \\ &- \frac{\Delta t}{2\Delta x}\left[\frac{\bar{z}^k_{i+1/2,j} + h_{i+1/2,j}}{1+\gamma^k_{i+1/2,j}\Delta t}(Fu^{k-1/2}_{i+1/2,j} + \Delta t\tau_x)\right. \\ &\left. - \frac{\bar{z}^k_{i-1/2,j} + h_{i-1/2,j}}{1+\gamma^k_{i-1/2,j}\Delta t}(Fu^{k-1/2}_{i-1/2,j} + \Delta t\tau_x)\right]. \end{aligned} \tag{40}$$

For each j, Eqs. (40) constitute a linear tridiagonal system with unknowns $z_{i,j}^{k+1/2}$. These systems are strictly diagonally dominant, symmetric, and with positive elements on the main diagonal and negative ones elsewhere. Therefore they all have a unique solution which can easily be determined by a direct method. Once the $z_{i,j}^{k+1/2}$ are determined, the new values for $u_{i+1/2,j}^{k+1/2}$ can be evaluated explicitly by using the first equation of system (39).

Next, we can proceed to the second level of calculation by finite differencing the y-momentum and the continuity equations as

$$\begin{aligned}
(1+\gamma_{i,j+1/2}^{k+1/2}\Delta t)\, v_{i,j+1/2}^{k+1} &= Fv_{i,j+1/2}^{k} - g\frac{\Delta t}{\Delta y}(z_{i,j+1}^{k+1} - z_{i,j}^{k+1}) + \Delta t\tau_y \\
z_{i,j}^{k+1} &= z_{i,j}^{k+1/2} - \frac{\Delta t}{2\Delta x}\big[(\bar{z}_{i+1/2,j}^{k+1/2} + h_{i+1/2,j})\, u_{i+1/2,j}^{k+1/2} \\
&\quad - (\bar{z}_{i-1/2,j}^{k+1/2} + h_{i-1/2,j})\, u_{i-1/2,j}^{k+1/2}\big] \\
&\quad - \frac{\Delta t}{2\Delta y}\big[(\bar{z}_{i,j+1/2}^{k+1/2} + h_{i,j+1/2})\, v_{i,j+1/2}^{k+1} \\
&\quad - (\bar{z}_{i,j-1/2}^{k+1/2} + h_{i,j-1/2})\, v_{i,j-1/2}^{k+1}\big],
\end{aligned} \tag{41}$$

Elimination of $v_{i,j\pm 1/2}^{k+1}$ in (41) yields

$$\begin{aligned}
z_{i,j}^{k+1} &- g\frac{\Delta t^2}{2\Delta y^2}\left[\frac{\bar{z}_{i,j+1/2}^{k+1/2} + h_{i,j+1/2}}{1+\gamma_{i,j+1/2}^{k+1/2}\Delta t}(z_{i,j+1}^{k+1} - z_{i,j}^{k+1})\right. \\
&\left. - \frac{\bar{z}_{i,j-1/2}^{k+1/2} + h_{i,j-1/2}}{1+\gamma_{i,j-1/2}^{k+1/2}\Delta t}(z_{i,j}^{k+1} - z_{i,j-1}^{k+1})\right] \\
&= z_{i,j}^{k+1/2} - \frac{\Delta t}{2\Delta x}\big[(\bar{z}_{i+1/2,j}^{k+1/2} + h_{i+1/2,j})\, u_{i+1/2,j}^{k+1/2} \\
&\quad - (\bar{z}_{i-1/2,j}^{k+1/2} + h_{i-1/2,j})\, u_{i-1/2,j}^{k+1/2}\big] \\
&\quad - \frac{\Delta t}{2\Delta y}\left[\frac{\bar{z}_{i,j+1/2}^{k+1/2} + h_{i,j+1/2}}{1+\gamma_{i,j+1/2}^{k+1/2}\Delta t}(Fv_{i,j+1/2}^{k} + \Delta t\tau_y)\right. \\
&\quad \left. - \frac{\bar{z}_{i,j-1/2}^{k+1/2} + h_{i,j-1/2}}{1+\gamma_{i,j-1/2}^{k+1/2}\Delta t}(Fv_{i,j-1/2}^{k} + \Delta t\tau_y)\right].
\end{aligned} \tag{42}$$

For each i, Eqs. (42) constitute a linear tridiagonal system with unknowns $z_{i,j}^{k+1}$. These systems are also symmetric, strictly diagonally dominant, and with positive elements on the main diagonal and negative ones elsewhere. Therefore their unique solution can easily be determined by a direct method. Once the $z_{i,j}^{k+1}$ are determined, the new values for $v_{i,j+1/2}^{k+1}$ can be evaluated explicitly by using the first equation of system (41).

It is worthwhile to point out that, if the computational region is divided into $n \times m$ finite difference cells, Eqs. (12) yield a linear, 5-diagonal system of $n \times m$ equations in $n \times m$ unknowns. While, if the alternating direction formulation is used, then, at each time step, we have to solve m tridiagonal systems of n equations (40) in n unknowns in the first level, and n tridiagonal systems of m equations (42) in m unknowns in the second level. On a vector computer these systems can be easily solved simultaneously by a vectorized direct method.

Note, finally, that both the semi-implicit method of Section 3 and the ADI method described above are mass conservative. Moreover, when the Eulerian–Lagrangian approach is used the new fluid velocity at any point (i, j) is directly related to the same quantity at the previous time step in $(i-a, j-b)$. Consequently, a more appropriate explicit evaluation of the bottom stress coefficient will be performed at $(i-a, j-b)$.

8. Computational Results

In order to illustrate some computational aspects of the methods described above, consider here a closed rectangular basin, of constant depth $h = 0.5\ m$, whose length, in the x direction is 6000 m, and whose width, in the y-direction is 3000 m. The basin is crossed centrally along the x direction by a deeper channel, which is closed at the right end and is open at the left end. The channel width is 300 m, while its depth, measured from the undisturbed water surface, is $h = 5$ m (see Fig. 4). The remaining flow parameters are $g = 9.81$, $\tau_x = \tau_y = 0$, and $C_z = 80$. At the basin boundaries the normal velocity is set equal to zero everywhere with the only exception being at the left end of the channel, where an M_2 tide of a 12 h period and 0.4 m amplitude is specified.

The flow domain is then divided into 40×20 finite difference cells of equal sides $\Delta x = \Delta y = 150$ m. The simulated tidal circulation begins with all water masses at rest. A dynamical steady state is reached after approximately ten complete tidal cycles when the numerical solution becomes almost perodical. Each method gives

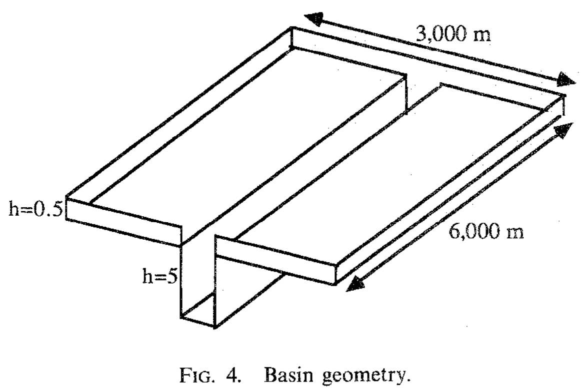

FIG. 4. Basin geometry.

a velocity field whose components at times $t = 108$ and $t = 120$ h agree to four decimal places.

Using a time subdivision $\tau = 30$ s, the numerical solution has been calculated with the semi-implicit method of Section 3 using $\Delta t = 60$ and $\Delta t = 360$ s. A comparison on the velocity at the $t = 120$ h showed a slight difference. The above calculations were then repeated with the alternating direction semi-implicit method of Section 8, again using $\Delta t = 60$ and $\Delta t = 360$ s, and again, a small difference of the velocity field was observed at $t = 120$ h. But, while the results obtained by the two methods with $\Delta t = 360$ had a difference within 10%, an excellent agreement was achieved when $\Delta t = 60$ was used. In this latter case the numerical results differed by no more than 1%. The simulation of one tidal cycle using time step $\Delta t = 360$, required 3.7 s of CPU time on a CRAY X-MP/48 by the first method, and only 1.4 s when the ADI method was used. Although the specific FORTRAN programs, for clarity, dit not use diagonal storage of matrices, each internal loop was naturally vectorized by the Cray Fortran Translator, and high performance was achieved.

Finally, in order to obtain a numerical solution with greater details, a finer grid of 60×120 cells of equal sides $\Delta x = \Delta y = 50$ m was used together with a time step $\Delta t = 60$ s and a time subdivision $\tau = 15$ s. For this example the average Courant number in the deep channel is then $C_r = 8$. The tidal circulation was again simulated for ten tidal cycles using both methods. The simulation of one tidal cycle required 79 s of CRAY CPU time by the semi-implicit method of Section 3, and 32 s by the ADI method. The numerical results obtained by the two methods at $t = 120$ h agree within a difference of 1%. Figures 5 and 6 show the resulting velocity field obtained at the 10th cycle 2 h before high water and 2 h before low water, respectively. These latter results also confirmed the validity of the results obtained by the two methods on the coarse mesh when $\Delta t = 60$ s was used. (The

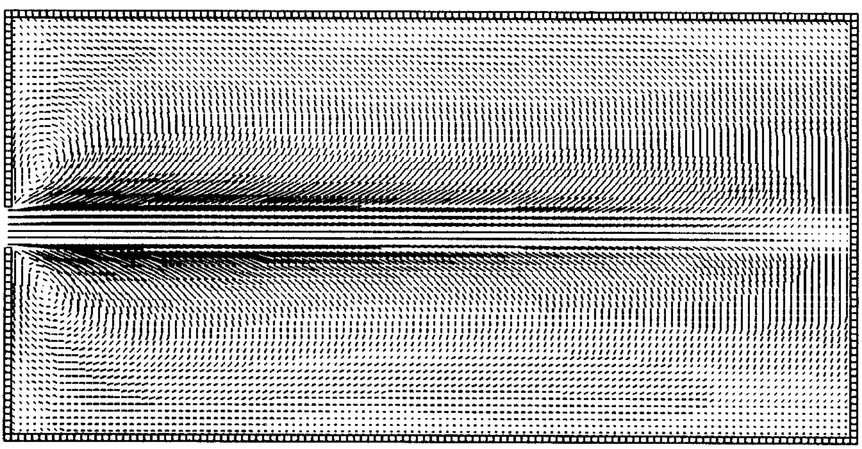

FIG. 5. Tidal circulation two hours before high water.

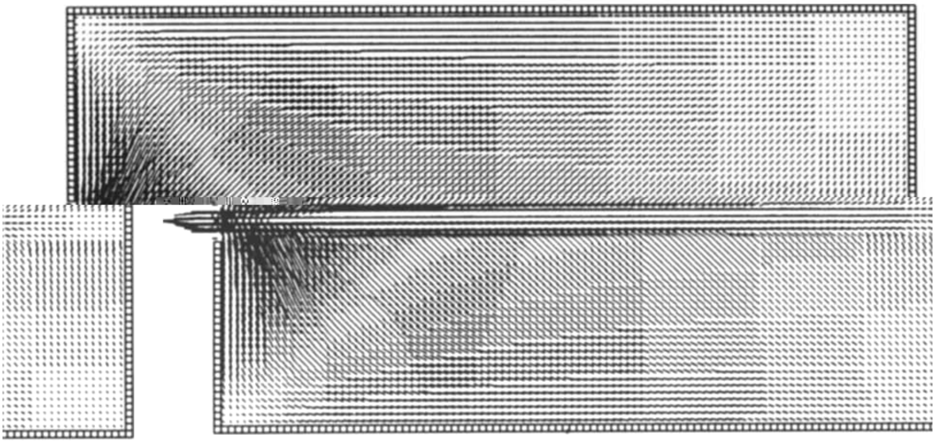

FIG. 6. Tidal circulation two hours before low water.

An application of a non-conservative form of the semi-implicit Eulerian–Lagrangian method has been made for simulating the tidal circulation in the Mar Piccolo, an embayment in the Gulf of Taranto, Italy (see Ref. [5]). On such a problem the numerical results have been very satisfactory hence confirming the method to be reliable, fast, accurate, and unconditionally stable even when the Coriolis forces are included in the model.

9. CONCLUSION

Some semi-implicit finite difference techniques for the shallow water equations have been presented and analyzed. A minimal degree of implicitness has been chosen in such a fashion that the stability of the method does not depend upon the wave celerity. The resulting linear system to be solved at each time step is simple, symmetric, 5-diagonal, and positive definite. Thus a fast preconditioned conjugate gradient method becomes suitable for determining uniquely the numerical slution at a reduced computational cost. Eulerian–Lagrangian or explicit upwind differencing can be used for the convective terms. When the Eulerian–Lagrangian formulation is used, the resulting algorithm is shown to be unconditionally stable; consequently, higher accuracy can be achieved since the artificial viscosity can be brought under control with the same time step size by reducing the spatial increments. Further improvements in the accuracy can be achieved by combining the present method with higher order fomulas (see, e.g., [13]) and/or with multigrid techniques. When the fluid is expected to have low velocity throughout, the stability condition (26) is not too restrictive, and the simpler upwind formulation is then to be preferred since, at each time step, a smaller computational effort is required by this method.

When applicable, the alternating direction semi-implicit algorithm can be combined with the present method to speed up the computations. In this case, at each time step, the implicitness of the discretization results into two sets of linear, tridiagonal systems whose unique solution can be easily determined by direct methods. Though, at present, a rigorous stability analysis of the alternating direction semi-implicit method is not available, several computational experiments have indicated that the stability condition given in Section 4 apply to this method as well. Nevertheless the numerical solution obtained with ADI techniques may become directionally dependent, and hence quite inaccurate, when large time steps are used (see Ref. [1] for details).

Finally, the methods described above are mass conservative and are so devised that they can be easily vectorized for an efficient implementation on modern, high speed vector computers.

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